and corresponding equation for the Green's function in the form

$$L_{mk}G_{kn} = -\delta_{mn} \tag{3}$$

The most popular method of solution of boundary-value problems like the foregoing is founded on Green's function. With its help, the moment on the nth support has the value

$$M_n = G_{nk}G_k \tag{4}$$

To modify this method and to change the finite-difference operator **L** and boundary values (2) into single operator **R** or to change the boundary-value problem (1) into the algebraic equation

$$R_{mn}M_n = -G_m \tag{5}$$

the operator **L** will be expanded. The eigenvalues can be found from the equation $|\mathbf{I} - \lambda\mathbf{L}| = 0$ or from the equivalent equation

$$[\delta_{m,n} - \lambda_k(\delta_{m,n+1} + 4\delta_{m,n} + \delta_{m,n-1})]\sin(k\pi m/N) = 0 \tag{6}$$

where

$$\lambda_k = \frac{1}{2[2+\cos(k\pi/N)]}$$

and $N - 1$ is the number of rows and columns in matrix **L**. But it is easy to prove that $\sin(k\pi n/N)$ represents the components of the eigenvectors of **L**. To carry out the normalization, these quantities must be multiplied by a constant factor, and then one gets the "projection tensors"

$$P_{m,n;k} = \frac{\sin\dfrac{k\pi m}{N}\sin\dfrac{k\pi n}{N}}{\sum\limits_{n=0}^{N}\sin^2\left(\dfrac{k\pi n}{N}\right)} \tag{7}$$

Thus one can write the operator **R**, which substitutes for (2) in expanded form,

$$\mathbf{R} = \frac{1}{\lambda_k}\mathbf{P}_k \tag{8}$$

and for Eq. (5) as follows:

$$(1/\lambda_k)P_{m,n;k}M_n = -G_m \tag{9}$$

From (9), with help of the tensor property of **P**,

$$\mathbf{P}_l\mathbf{P}_k = \delta_{lk} \tag{10}$$

one gets, after multiplication from the left by $\mathbf{P}_l$, the explicit solution of the "three-moments equation:"

$$M_k = -\lambda_l P_{k,n;l}G_n \tag{11}$$

The comparison of (11) with (4) yields Green's function

$$G_{nm} = -\lambda_l P_{nm;l} \qquad l = 0,1,\ldots N \tag{12}$$

The comparison of (12) and (8) reveals that **R** and **G** are commuting operators. This property distinguishes the operators **R** and (2). After summation, it is possible to write Green's function (12) in the form

$$W_{nk} = -\lambda_l P_{n,k;l} = \frac{(-1)^{n+k}}{2\sinh\sigma}\left\{\sinh|n-k|\sigma - \sinh(n+k)\sigma + \frac{2\sinh n\sigma\sinh k\sigma}{\tanh N\sigma}\right\} \tag{13}$$

where $\sinh\sigma = 3^{1/2}$. Then the explicit solution (11) takes more compact form:

$$M_n = W_{nm}G_m \tag{14}$$

Adding to (14) regular part, one can write the solution of the boundary value problem

$$M_{n+1} + 4M_n + M_{n-1} = -G_n$$
$$M_0 = M_0 \qquad M_N = M_N \qquad n = 0,1,\ldots N \tag{15}$$

as follows:

$$M_n = W_{nk}G_k - (-1)^{N+n}(\sinh n\sigma/\sinh N\sigma) \times [(-1)^N e^{N\sigma}M_0 - M_N] + (-1)^n e^{n\sigma}M_0 \tag{16}$$

To facilitate insight into the structure of Green's function (12) and (13), one can take, as in the previous case, $N = 3$. Then $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{1}{3}$, and

$$G_{nm} = -\left(\frac{1}{5}\frac{\sin\frac{\pi m}{3}\sin\frac{\pi n}{3}}{\sum\limits_{n=0}^{3}\sin^2\left(\frac{\pi n}{3}\right)} + \frac{1}{3}\frac{\sin\frac{2\pi m}{3}\sin\frac{2\pi n}{3}}{\sum\limits_{n=0}^{3}\sin^2\left(\frac{2\pi n}{3}\right)}\right)$$

For particular numbers

$$\mathbf{G} = -\begin{vmatrix}\frac{1}{10}+\frac{1}{6}, & \frac{1}{10}-\frac{1}{6}\\ \frac{1}{10}-\frac{1}{6}, & \frac{1}{10}+\frac{1}{6}\end{vmatrix}$$

Assuming, for example, $N = 3$ and the load terms, $G_1 = G_2 = \frac{1}{2}pl^2$, one obtains from (15) the following expression for moments:

$$M_1 = M_2 = \frac{1}{2\sinh\sigma}\left\{-\sinh\sigma + 3\sinh 3\sigma - \sinh 4\sigma + \frac{2\sinh\sigma}{\tanh 3\sigma}\cdot[\sinh\sigma - \sinh 2\sigma + \sinh 3\sigma]\right\}\frac{1}{2}pl^2 = -\frac{1}{10}pl^2$$

in agreement with previous results.

Test Time in a 1.5-Inch-Diameter High-Stagnation-Enthalpy Shock Tube

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In a recent article, Hoshizaki[1] questioned the validity of heat transfer measurements[2] made at high-stagnation-enthalpy conditions in a 1.5-in.-diam shock tube. The point in question is the available testing time in low pressure, high shock Mach number flow in a small diameter shock tube. The present note on test time measurements in the 1.5-in.-diam shock tube appears necessary to establish the validity of these measurements. Since the measurements proved to be quite accurate compared with Hoshizaki's theory and experimental results (as well as more recent data obtained by Rose and Stankevics[3] at the Avco Everett Laboratory), it would be expected that the test time was adequate.

The original heat transfer measurements relied on the heat transfer calorimeter gage to indicate test time. Camm and Rose[4] recently found that calorimeter measurements indicate steady test times that are longer than those determined by other methods. Apparently, the heat transfer is not changed by the change in composition of the flow at the interface. Since heat transfer is proportional mainly to the enthapy and density rather than temperature or composition, Camm and Rose suggest that heat transfer is unchanged by trace im-

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purities present in the interface region. Thus, it would be expected that the 15- to 25-μsec test time reported in Ref. 2 is greater than would be found by other means of indicating test time.

Figure 1 shows results of measurements of test time for the 1.5-in.-diam shock tube made with photomultipliers and electron collector probes. The photomultiplier was focused on the center of the tube through a series of collimated slits so that a spatial resolution of approximately 1 mm was obtained. The electron collector probe was a simple wire extended into the flow and biased with a positive voltage (+6 v above shock tube ground) so that electrons are attracted to the probe. Some uncertainty in test time exists in both of these measuring methods in that neither radiation nor ionization occurs directly at the shock front. The uncertainty due to the finite adjustment time should be of the order of 0.5 μsec at the higher shock Mach numbers (based on the extrapolation of ionization time measurements). For the photomultiplier measurements, shown in Fig. 1, below a shock Mach number of 15, radiation from impurities in the driven gas was sufficient to indicate the shock front. A nonequilibrium overshoot in radiation,[4] which lasts for an average of 2 μsec, is present in the photomultiplier traces. Thus, for $M_s \approx 24$ and an initial pressure of 250 μ Hg, equilibrium test times from 6 to 4 μsec are indicated. No change in the measured heat transfer rates of Ref. 2 was found when the data were re-evaluated using the forementioned results.

Hoshizaki's question was based on theoretical test times calculated from the analysis of Roshko.[5] Roshko's analysis has been improved by Camm and Rose,[4] and their results have been used to predict the shaded curves, shown for the different initial pressures, on Fig. 1. The upper limit of each curve is that predicted directly from theoretical considerations. Experimental measurements of shock tube test times at low shock Mach numbers reported previously[6] have been re-analyzed in the light of Camm and Rose's improved theory, and the data are found to follow closely the trends of the theory. However, it was found that the data are systematically lower than the predicted values. The lower limit of the shaded curves was determined from a faired curve through the data. As can be seen, the present test time measurements at $P_1 = 1$ and 0.5 mm Hg are in good agreement with the low shock Mach number measurements.[6] A boundary layer thickness parameter β enters directly into the determination of the curves shown on Fig. 1. The parameter β is computed from laminar boundary layer solutions obtained by Mirels.[7] The solutions of Mirels are for initial pressures of 0.76 and 7.6 mm Hg and a range of shock Mach numbers between 4 and 14. Outside this range of conditions an extrapolation formula is given. For the low initial pressure conditions ($p_1 \leq 0.25$ mm Hg), it appears that the value of β is lower than would be predicted by extrapolation. This may explain the low values of test time computed by Hoshizaki (2 to 3 μsec).

The values of test time shown in Fig. 1 are at greater distances down the shock tube than the original data. The data below a Mach number of 15 were at a station 22.6 ft from the diaphragm, and the high Mach number data were taken at a station 23.7 ft from the diaphragm. The data of Ref. 2 were taken at a station 16 ft from the diaphragm. A reduction of 5% in test time at $p_1 = 0.25$ mm Hg would be predicted from the theoretical calculations if the length of the shock tube were reduced from 23.7 to 16 ft. (The theoretical curves of Fig. 1 are for a length of 23.7 ft.) Thus, the effect of the shorter length is negligible for this case. For greater initial pressures the reduction in test time increases.

According to the theoretical analysis of low-pressure shock tube test time, the time should increase as the square of the tube diameter. The low Mach number data of Ref. 6 were taken in a 3-in.-diam tube. The agreement of the $p_1 = 1$ and 0.5 mm Hg data with the predicted curves indicates that the scaling is reasonable. On the other hand, the test

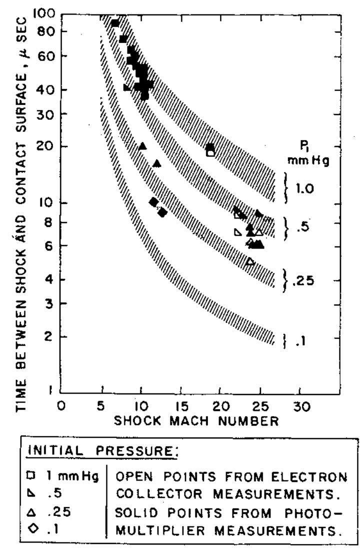

Fig. 1 Test time measurements in 1.5-in.-diam shock tube compared with predicted test time

times obtained by Camm and Rose[4] for a 6-in.-diam shock tube fall far below theoretical values. (Similar results have been obtained for a 6.5-in.-diam shock tube at the Avco Research and Advanced Development Division.) The 6-in.-diam tube had a 1.328-in.-diam driver and expands out (after the diaphragm) to the 6-in. diam. Apparently the expansion process greatly reduces the test time. The larger tube gives longer test times than the 1.5-in.-diam tube, but the gain is far less than expected from theoretical considerations.

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Comment on the Soap-Film Paradox

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In a previous paper,[1] Gellatly discussed a seeming contradiction in the calculation of the equilibrium configuration of a soap film hanging from a horizontal ring. The paradox may be seen in an even simpler configuration, namely, a soap film hanging in a vertical, plane wire frame. Gellatly

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